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1991 J. Phys. A: Math. Gen. 24 L1105

(http://iopscience.iop.org/0305-4470/24/18/007)

J. Phys. A: Math. Gen. 24 (1991) L1105–L1110. Printed in the UK

LETTER TO THE EDITOR

The dimension of turbulence

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Received 11 March 1991

Abstract. We suggest a scenario for turbulence where the fractal dimension D increases with Reynolds number: $D=2$ at the onset of turbulence, and $D \to 3$ at large Reynolds number. The picture is based on a new random-cascade model where the length scales of active eddies are randomly chosen from a probability distribution $P(r)$ of length-scale ratios. Exact expressions for the exponents associated with the velocity field are derived for the distributions $P(r)=(\gamma+1)r^{\gamma}$. Our picture provides a quantitative explanation for recent measurements of pipe and grid flow at the onset of turbulence.

Perhaps the most intriguing phenomenon in turbulence is the occurrence of an energy cascade from an initial length scale l_0, to smaller and smaller length scales. To describe this behaviour, Kolmogorov [1] proposed a scaling theory for fully developed turbulence with a uniform energy transfer ε. Based on the assumption that turbulence is space-filling†, the change in velocity $v(l)$ over an 'active' eddy of size l was shown to scale,

$$v(l) \sim l^h \tag{1}$$

with exponent $h=\frac{1}{3}$. However, as pointed out by Mandelbrot [2], turbulence is not necessarily space-filling—the active eddies may decay in volume, leaving a fractal set of singularities with a fractal dimension $D<3$.

Considerations like those above have led to various scaling theories for turbulence. Among these is the so-called β-model [3], where one posits the (arbitrary) length scales

$$l_n = l_0 2^{-n} \tag{2}$$

and defines β as the factor by which the volume of the active eddies is contracted when energy is transferred from scale l_n to scale l_{n+1}. The number N_n of eddies at scale l_n is $N_n=\beta^n l_0^3/l_n^3$ and this relates β to the fractal dimension D,

$$\log_2 \beta = D-3 \tag{3}$$

where D is defined by

$$N_n \sim l_n^{-D}. \tag{4}$$

The energy transfer is $\varepsilon=\beta^n v_n^3/l_n$, where $v_n \equiv v(l_n)$ is the velocity difference over an active eddy at 'level' n. Assuming ε to be constant, it follows that

$$h=\tfrac{1}{3}(D-2). \tag{5}$$

† Mathematically speaking, turbulence here refers to the set of singular points of the Navier–Stokes equations.

0305-4470/91/181105+06$03.50

Recently, Tong and Goldburg [4] have measured the characteristic velocity difference $v(l)$ by photon-correlation homodyne spectroscopy and determined the exponent h for the turbulent flow. The measurements were carried out for both a pipe and grid flow, in the regime of small Reynolds number at the onset of turbulence. The resulting values of h as function of the Reynolds number are shown in figure 1. h increases from the value $h=0$ obtained at the critical Reynolds number signaling the onset of turbulence, $\mathrm{Re}=\mathrm{Re}_c$, to the Kolmogorov value $h=\frac{1}{3}$ at large Reynolds number. In terms of the dimension D (equation (5)), this corresponds to an increase from $D=2$ at the $\mathrm{Re}=\mathrm{Re}_c$, to $D=3$ for fully developed turbulence—the turbulence becomes more and more space filling. For the β-model, the change of D from 2 to 3 corresponds to an increase of β from $\frac{1}{2}$ to 1. It is not clear, though, how and why β should depend on the Reynolds number.

In this letter we propose a new cascade model with the aim to understand the changing dimensionality of turbulence. Our picture arises from a natural concern regarding the β-model; namely, its reliance on a fixed and arbitrary length-scale ratio $r \equiv l_n/l_{n-1}=\frac{1}{2}$ (cf equation (2)) between an eddy and its 'mother' eddy. What happens if this ratio is not constant—if r can take on different values, $r(i)$, for each eddy i? To analyse this situation, we study a model where every active eddy splits into m smaller eddies with length scales chosen according to a probability distribution $P(r)$ of length-scale ratios r (given by the dynamics). We shall show that D and h increase with m; the greater the splitting of active eddies, the more turbulence fills space. Therefore, one scenario at the onset of turbulence is that m increases with increasing Reynolds number. As Re increases, the dissipation scale l_d decreases, and this allows the formation of smaller eddies and thereby permits a larger value of m. The basic idea is a maximum-entropy principle: *An active eddy splits into as many active eddies as possible, given the dynamically fixed distribution* $P(r)$. We shall elaborate on this point and find a relation between m and Re.

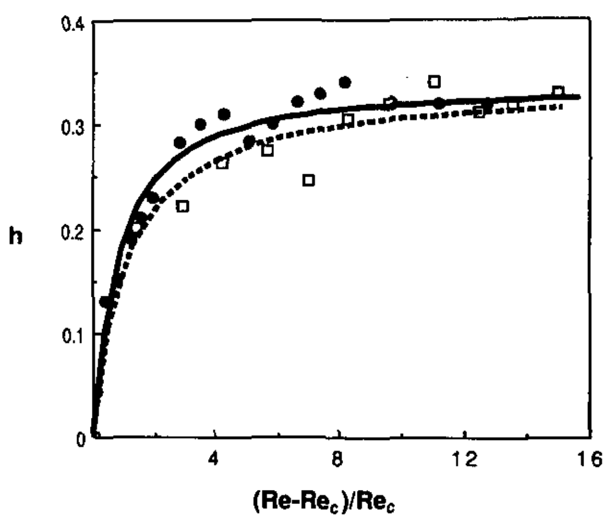

Figure 1. Velocity-field exponent h obtained from homodyne spectroscopy for various values of the Reynolds number Re (reference [4]). (□) Pipe flow, $\mathrm{Re}_c=2160$ (●) Grid flow, $\mathrm{Re}_c=263$. Also shown are the theoretical $h(\mathrm{Re})$ curves for $\gamma=\frac{1}{4}$ ($m_c=4.1809$) - - -), and $\gamma=\frac{1}{2}$ ($m_c=3.6946$) (——).

Assume that every eddy of size $l(i)$ splits into m eddies of sizes $l_1(i), l_2(i), \ldots, l_m(i)$. The length-scale ratios $r_j(i) \equiv l_j(i)/l(i) (j = 1, 2, \ldots, m)$ are chosen independently and at random, according to a given probability distribution $P(r)$. For $P(r) = \delta(r - \frac{1}{2})$, we have the β-model (with $\beta = m/8$). To assure a decreasing volume, only a choice of $r_1(i), r_2(i), \ldots, r_m(i)$ within the supersphere S given by

$$r_1^3(i) + r_2^3(i) + \ldots + r_m^3(i) \leqslant 1 \tag{6}$$

is accepted†. The probability density $P(\boldsymbol{r})$ for a set $\boldsymbol{r} = (r_1, r_2, \ldots, r_m)$ of length-scale ratios is

$$P(\boldsymbol{r}) = \frac{\Pi_{j=1}^m P(r_j)}{\int_S \Pi_{j=1}^m P(r_j)\, \mathrm{d}r_j}. \tag{7}$$

The number of eddies at level n is $N_n = m^n$. For random length scales, the fractal dimension D is no longer given by equation (4). Rather, we study the partition function

$$Z_n(q) \equiv l_0^{-q} \left\langle \sum_{i=1}^{m^n} l_n^q(i) \right\rangle \tag{8}$$

where $\langle\,\rangle$ denotes the average over all configurations of length scales. In the limit of large n this defines a 'free energy'

$$F(q) \equiv -\lim_{n\to\infty} n^{-1} \log_m Z_n(q). \tag{9}$$

For the β-model, $Z_n(q) = m^n 2^{-nq}$, and $F(q)$ is linear, $F(q) = (q - D)/D$ (here $D = \log_2 m$).

The dimension D is given implicitly by $F(D) = 0$. Since l_n has a multinomial (m-nomial) distribution in the factors $r_1, r_2, \ldots, r_m$, it follows that l_n^q has an m-nomial distribution in the factors $r_1^q, r_2^q, \ldots, r_m^q$. Therefore,

$$Z_n(q) = I^n(q) \tag{10}$$

where

$$I(q) \equiv \int_S (r_1^q + r_2^q + \ldots + r_m^q) P(\boldsymbol{r})\, \mathrm{d}^m \boldsymbol{r}. \tag{11}$$

By symmetry, (11) reduces to

$$I(q) = m \int_S r_1^q P(\boldsymbol{r})\, \mathrm{d}^m \boldsymbol{r}. \tag{12}$$

The free energy is by definition (cf equation (9)) $F(q) = -\log_m I(q)$, thus D is determined by‡

$$I(D) = 1. \tag{13}$$

$I(q)$ is a decreasing function of q. Moreover, $I(0) = m$, and by equation (6), $I(3) \leqslant 1$; we then have $0 \leqslant D \leqslant 3$. Note that $D = 0$ only when $m = 1$, and $D = 3$ only when $P(r) = \delta(r - m^{-1/3})$.

† The distribution $P(r)$ must be zero outside the unit interval and non-zero for at least one value of r below $m^{-1/3}$.

‡ In correspondence with the random β-model [5], one can include a variable m given by a probability distribution $p(m)$. In this case, D is obtained as the solution of $\Pi_m [I_m(q)]^{p(m)} = 1$, where $I_m(q)$ is the value of $I(q)$ for a specific value of m.

Although an entire spectrum of exponents $F(q)$ is necessary to fully describe the length-scale struture, the average behaviour of the velocity differences depends only on the dimension D. More precisely, the energy transfer from level n to level $n+1$ is

$$\varepsilon = l_0^{-3}\left\langle \sum_{i=1}^{m^n} l_n^3(i) v_n^3(i)/l_n(i) \right\rangle \tag{14}$$

and based on a constant value for ε, this leads to the scaling behaviour (1) with h again given by (5).

For the β-model, h is a simple function of m, $h(m)=\frac{1}{3}\log_2(m/4)$, $4 \leqslant m \leqslant 8$. For our class of models, $D(m)$ and therefore $h(m)$ are generally more complicated. Consider, for example, the distributions $P(r)=(\gamma+1)r^\gamma$, for $0<r\leqslant 1$, $\gamma>-1$. Then,

$$I(q) = m \int_S r_1^q (r_1 \ldots r_m)^\gamma \, \mathrm{d}^m \boldsymbol{r} \bigg(\int_S (r_1 \ldots r_m)^\gamma \, \mathrm{d}^m \boldsymbol{r} \bigg) \tag{15}$$

which can be integrated to yield†

$$I(q) = m \frac{\Gamma[\frac{1}{3}(q+\gamma+1)]\Gamma[\frac{1}{3}m(\gamma+1)+1]}{\Gamma[\frac{1}{3}(\gamma+1)]\Gamma[\frac{1}{3}(q+m(\gamma+1))+1]} \tag{16}$$

where $\Gamma(x)$ is the gamma function.

D is determined by solving equation (13). Figure 2 shows D for the cases $\gamma=0, 1$ and $1 \leqslant m \leqslant 10$ (also shown in $D(m)$ for the β-model). Since increased values of γ give more weight to large values of r, the fractal dimension will likewise increase. Furthermore, the dimension D increases with m, and at large m, D approaches the value 3 with an asymptotic behaviour

$$D \simeq 3 - \frac{9}{(\gamma+1)(m \ln m)}. \tag{17}$$

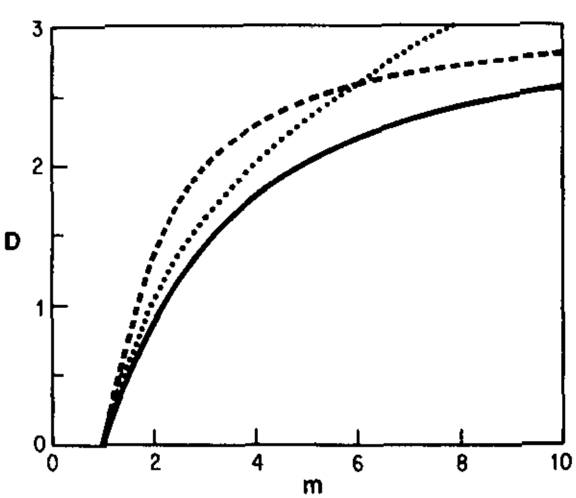

Figure 2. Fractal dimension D determined for various values of m. (——): $P(r)=1$. (– – –): $P(r)=2r$. (· · ·): β-model.

† The integrals are facilitated by a change of variables from r_j to $u_j \equiv r_j/x_{j-1}$ ($j=1,2,\ldots,m$), where $x_j \equiv (1-r_1^3-\ldots-r_j^3)^{1/3}$, $x_0=1$. Note that $x_j = x_{j-1}(1-u_j^3)^{1/3}$.

To connect the above analysis to experiment we must find the relation between Re and m. First notice that the Reynolds number scales with the dissipation scale l_d [3],

$$\mathrm{Re} \sim l_d^{-(h+1)}. \tag{18}$$

l_d sets the smallest possible length scale for an active eddy, and (18) follows from comparing the 'turnover' time l_n/v_n with the dissipation time $l_n^2/\nu \sim l_n^2\,\mathrm{Re}$ (ν is the viscosity). Now, to relate l_d to m we invoke the 'maximum-entropy' principle: An active eddy splits into as many active eddies as possibly allowed by $P(r)$. In fact, $P(r)$ places severe constraints on m. One obvious constraint is that m cannot grow larger than $r_{\min}^{-3}$, where $r_{\min} = \inf\{r|P(r) \neq 0\}$ (for the β-model, $r_{\min} = \frac{1}{2}$). But even when $r_{\min} = 0$, m is effectively bounded because large values of m create length scales almost all of which are smaller than l_d, in contradiction with the requirement that active eddies be formed. To fulfil this requirement, the probability of creating m active eddies must be of order 1,

$$\int_{S(l_d)} P(\boldsymbol{r})\,\mathrm{d}^m\boldsymbol{r} = \mathcal{O}(1) \tag{19}$$

where $S(l_d)$ is the part of S with $r_j > l_d$ for all j. $m(\mathrm{Re})$ is then defined as the largest value of m for which condition (19) is satisfied. To make the definition rigorous, $\mathcal{O}(1)$ must be replaced by a constant (say, $\frac{1}{2}$).

For the β-model, equation (19) gives a sharp transition from laminar flow to space-filling turbulent flow ($m = 8$) at a critical Reynolds number for which $l_d = \frac{1}{2}$. For our power-law distributions, (19) gives the scaling relation

$$m \sim l_d^{-(\gamma+1)}. \tag{20}$$

Using this and the scaling law (18), we end up with

$$\mathrm{Re} \sim m^{(h+1)/(\gamma+1)}. \tag{21}$$

To compare our theory with experiment (figure 1), we make m a real variable, and determine the proportionality factor in (21) by setting $\mathrm{Re} = \mathrm{Re}_c$ at the value m_c defined by $D(m_c) = 2$. The curves in figure 1 show the $h(\mathrm{Re})$ functions obtained from equations (5), (13), (16) and (21) for $\gamma = \frac{1}{4}$ and $\gamma = \frac{1}{2}$. (The significant variation of $h(\mathrm{Re})$ with γ stems from the explicit γ-dependence in equation (21).) We conclude that for both pipe and grid flows the data are well described by the theoretical h curves, in the γ regime $[\frac{1}{4}, \frac{1}{2}]$. No relation between the form of $P(r)$ and the Navier–Stokes equations has been established thus far. It may be possible to obtain such a relation through computational studies of the temporal development of bursts [6], and we urge studies in that direction.

Note that the 'two-point' homodyne experiments differ from the usual 'one-point' Doppler velocimetry which measures the local velocity [4]. Adopting the frozen-turbulence assumption that velocity changes in time translate to velocity differences in space, a velocity-difference distribution can be obtained from the latter experiment. This differs from that obtained through homodyne spectroscopy, as the velocities will now be weighted by their occurrence. The exponent for the spatially-averaged moments $\overline{v^q(l)}$ of the velocity-difference distribution is denoted ζ_q,

$$\overline{v^q(l)} \sim l^{\zeta_q}. \tag{22}$$

For the β-model, the active eddies at level n are weighted by a factor β^n, and $\zeta_q = hq - \log_2 \beta$. In terms of the fractal dimension, we have from equations (3) and (5),

$$\zeta_q = \tfrac{1}{3}[(9-2q) - (3-q)D]. \tag{23}$$

The same expression is obtained in our class of models, where the moments of the velocity-difference distribution are given by

$$\overline{v^q(l)} = l_0^{-3} \sum_{l_n(i)\approx l} l_n^3(i) v_n^q(i). \tag{24}$$

By $l_n(i) \approx l$ we refer to a region of width l, around l. The choice affects only prefactors, not the exponents ζ_q. To obtain equation (23), differentiate the sum in (24) with respect to $l(v_n \sim l_n^h)$: $\sum_{l-dl<l_n(i)\leq l} l_n^{3+hq} = l^{3+hq} \sum_{l-dl<l_n(i)\leq l} 1$. The last sum is the derivative of the total number $N(l)$ of intervals larger than l. The result (23) for ζ_q follows because $N(l) \sim l^{-D}$. In contrast to the behaviour of h, we note that when q is less than 3, the exponent ζ_q decreases with D. Preliminary experimental data have confirmed this behaviour for $q=2$ [7].

In summary, we have analysed a random-cascade model of turbulence, where the length scales are determined from a probability distribution $P(r)$ of length-scale ratios. We have studied a class of models where this distribution is power-law and have found exact expressions for the fractal dimension D and the velocity-field exponents, h and ζ_q, of the turbulent flow. Our model suggests a scenario for turbulence where an increase in Reynolds number leads to an increase in the splitting of active eddies which, in turn, increases the degree to which turbulence fills space. This scenario provides a quantitative explanation for the experimentally observed increase of h and decrease of ζ_2 with increasing Reynolds number.

We thank W I Goldburg, J V Maher and H E Stanley for helpful discussions. The Center for Polymer Studies is supported by grants from the National Science Foundation and the Office of Naval Research. In addition, GH is supported by a DARPA/NASA fellowship in parallel processing, administered by the IACS, University of Maryland. PA acknowledges support from the Danish Natural Science Research Council and the Carlsberg Foundation.

References

[1] Kolmogorov A N 1941 *C. R. Acad. Sci. USSR* **30** 301
[2] Mandelbrot B B 1974 *J. Fluid Mech.* **62** 331
[3] Frisch U, Sulem P and Nelkin M 1978 *J. Fluid Mech.* **87** 719
[4] Tong P and Goldburg W I 1988 *Phys. Fluids* **31** 2841
[5] Benzi R, Paladin G, Parisi G and Vulpiani A 1984 *J. Phys. A: Math. Gen.* **17** 3521
[6] Jensen M H and Paladin G Private communication
[7] Pak H K and Goldburg W I Private communication